The Price of Doing Business: How Startup Costs Improve Government Treatment of Foreign Firms

Online Appendix

Proof of Proposition 1.

1.1 Consumer Demand

Recall that:

$$U = \left(1 - \sum_{j=1} w_j\right) \log v_0 + \sum_{j=1} w_j \log Q_j \quad \text{where: } Q_j \equiv \left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}}$$

If the price of a variety v in industry j is $p_j(v)$ and the quantity consumed is $q_j(v)$, then the budget constraint is:

$$\sum_{j=0} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv \leq R$$

where R is the aggregate revenue.

The Lagrangian is:

$$\mathcal{L} = \left(1 - \sum_{j=1} w_j\right) \log v_0 + \sum_{j=1} w_j \log Q_j + \lambda \left[R - \sum_{j=0} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv\right]$$

Then constrained optimization for industry $j \in \{1, \ldots, J\}$ and product v' yields:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_j(v')} &= \frac{w_j}{Q_j}\left(\frac{\sigma}{\sigma-1}\right)\left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{\frac{\sigma}{\sigma-1}-1} \left(\frac{\sigma-1}{\sigma}\right) q_j(v')^{\frac{\sigma-1}{\sigma}-1} - \lambda p_j(v') \\
&= w_j \left[\int_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv\right]^{-1} q_j(v')^{-\frac{1}{\sigma}} - \lambda p_j(v') \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= R - \sum_{j=0} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv \\
\lambda &\geq 0 \\
\lambda \frac{\partial \mathcal{L}}{\partial \lambda} &= \lambda \left[R - \sum_{j=0} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv\right] = 0
\end{aligned}$$

Note that if $\lambda = 0$, then $\frac{\partial \mathcal{L}}{\partial q_j(v')} > 0$ for all levels of consumption. So it must be true that $\lambda > 0$ and $R - \sum_{j=0} \int_{v \in V_j} p_j(v)\, q_j(v)\, dv = 0$. (That is, the budget constraint is binding.)

Additionally, note that for any industry j and for any goods v, $v' \in V_j$, $\frac{\partial \mathcal{L}}{\partial q_j(v)} = \frac{\partial \mathcal{L}}{\partial q_j(v')} = 0$ iff:

$$\begin{aligned}
\lambda &= w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v)^{-\frac{1}{\sigma}} p_j(v)^{-1} = w_j \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{-1} q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} \\
&\Leftrightarrow \quad q_j(v)^{-\frac{1}{\sigma}} = q_j(v')^{-\frac{1}{\sigma}} p_j(v')^{-1} p_j(v) \\
&\Leftrightarrow \quad q_j(v)^{\frac{\sigma-1}{\sigma}} = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} p_j(v)^{1-\sigma} \\
&\Leftrightarrow \quad \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv = q_j(v')^{\frac{\sigma-1}{\sigma}} p_j(v')^{\sigma-1} \int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \\
&\Leftrightarrow \quad q_j(v')^{\frac{\sigma-1}{\sigma}} = p_j(v')^{1-\sigma} \int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{-1} \\
&\Leftrightarrow \quad q_j(v') = p_j(v')^{-\sigma} Q_j P_j^{\sigma} \quad \text{where: } P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}
\end{aligned}$$

So the demand function for any variety v in any industry j is:

$$q_j(v) = p_j(v)^{-\sigma} Q_j P_j^{\sigma}$$

where:

$$Q_j \equiv \left[\int\limits_{v \in V_j} q_j(v)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \quad \text{and} \quad P_j \equiv \left[\int\limits_{v \in V_j} p_j(v)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}$$

1.2 Production

Let the wage be 1, which represents one unit of the numeraire good, which is the unique good produced in industry $j = 0$. To simplify notation, we suppress industry notation. Since in a model of monopolistic competition, each firm produces a unique good, we can refer to each good by the productivity of the firm that produces it. That is, if a firm of type φ' produces good v', we can use the terms $p(v')$ and $p(\varphi')$ interchangably.

To make an output of q units, a domestic firm must use labor, $\frac{q}{\varphi}$, where φ is the firm's productivity. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by domestic firms is:

$$\pi_d(\varphi) = p_d(\varphi) q_d(\varphi) - \left[\frac{q_d(\varphi)}{\varphi} + c \right] = p_d(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_d(\varphi)^{-\sigma} QP^{\sigma}}{\varphi} + c \right]$$

Profit maximization by domestic firms yields:

$$\begin{aligned}
\frac{\partial \pi_d(\varphi)}{\partial p_d} &= (1-\sigma) p_d(\varphi)^{-\sigma} QP^{\sigma} + \sigma \left[\frac{p_d(\varphi)^{-\sigma-1} QP^{\sigma}}{\varphi} \right] = 0 \\
&\Leftrightarrow \quad p_d(\varphi) = \frac{\rho}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma-1}
\end{aligned}$$

This yields revenue:

$$r_d(\varphi) = \left(\frac{\varphi}{\rho} \right)^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_d(\varphi) &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma} - \left[\frac{\left(\frac{\varphi}{\rho}\right)^{\sigma} QP^{\sigma}}{\varphi} + c\right] \\ &= \left(\frac{\varphi}{\rho}\right)^{\sigma-1} QP^{\sigma}\left(1-\frac{1}{\rho}\right) - c = \frac{r_d(\varphi)}{\sigma} - c\end{aligned}$$

To make an output of q units, a foreign firm must use labor, $\frac{q(1+\tau)}{\varphi}$, where φ is the firm's productivity and τ is the government taking rate. Let c denote the fixed per period cost of production (in terms of labor). By using the demand function from above (and suppressing the industry notation), we can see that the per period profit from production by foreign firms is:

$$\pi_f(\varphi) = p_f(\varphi)\, q_f(\varphi) - \left[\frac{q_f(\varphi)(1+\tau)}{\varphi} + c\right] = p_f(\varphi)^{1-\sigma} QP^{\sigma} - \left[\frac{p_f(\varphi)^{-\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right]$$

Profit maximization by foreign firms yields:

$$\begin{aligned}\frac{\partial \pi_f(\varphi)}{\partial p_f} &= (1-\sigma)\, p_f(\varphi)^{-\sigma} QP^{\sigma} + \sigma\left[\frac{p_f(\varphi)^{-\sigma-1} QP^{\sigma}(1+\tau)}{\varphi}\right] = 0 \\ &\Leftrightarrow \quad p_f(\varphi) = \frac{\rho(1+\tau)}{\varphi} \quad \text{where: } \rho = \frac{\sigma}{\sigma-1}\end{aligned}$$

This yields revenue:

$$r_f(\varphi) = \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}$$

and profit:

$$\begin{aligned}\pi_f(\varphi) &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma} - \left[\frac{\left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma} QP^{\sigma}(1+\tau)}{\varphi} + c\right] \\ &= \left[\frac{\varphi}{\rho(1+\tau)}\right]^{\sigma-1} QP^{\sigma}\left(1-\frac{1}{\rho}\right) - c = \frac{r_f(\varphi)}{\sigma} - c\end{aligned}$$

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Therefore, conditional on being in the market, optimal production by domestic firms in industry j yields:

$$p^*_{dj}(\varphi) = \frac{\rho}{\varphi} \quad \text{and} \quad q^*_{dj}(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma} Q_j P_j^{\sigma} \quad \text{and} \quad r^*_{dj}(\varphi) = \left(\frac{\varphi}{\rho}\right)^{\sigma-1} Q_j P_j^{\sigma}$$

This yields domestic firm profit in industry j:

$$\pi^*_{dj}(\varphi) = \phi_{dj}\varphi^{\sigma-1} - c \qquad \text{where: } \phi_{dj} \equiv \frac{Q_j P_j^{\sigma}}{\sigma\rho^{\sigma-1}}$$

And optimal production by foreign firms in industry j yields:

$$p^*_{fj}(\varphi) = \frac{\rho(1+\tau_j)}{\varphi} \quad \text{and} \quad q^*_{fj}(\varphi) = \left[\frac{\varphi}{\rho(1+\tau_j)}\right]^{\sigma} Q_j P_j^{\sigma} \quad \text{and} \quad r^*_{fj}(\varphi) = \left[\frac{\varphi}{\rho(1+\tau_j)}\right]^{\sigma-1} Q_j P_j^{\sigma}$$

This yields foreign firm profit in industry j:

$$\pi^*_{fj}(\varphi) = \phi_{fj}\varphi^{\sigma-1} - c \qquad \text{where: } \phi_{fj} \equiv \frac{Q_j P_j^{\sigma}}{\sigma\rho^{\sigma-1}(1+\tau_j)^{\sigma-1}}$$

1.3 Firm Entry and Exit

We now return to suppressing industry subscripts to simplify notation. First, note the following properties of the profit functions for $i \in \{d, f\}$:

$$\lim_{\varphi \to 0} \pi_i^* (\varphi) = -c < 0 \quad \text{and} \quad \lim_{\varphi \to \infty} \pi_i^* (\varphi) = \infty > 0$$

$$\text{and} \quad \frac{\partial \pi_i^* (\varphi)}{\partial \varphi} = (\sigma - 1) \phi_i \varphi^{\sigma - 2} > 0$$

So each function $\pi_i^* (\varphi)$ is partially invertible: for any $y > -c$, there exists a unique $\varphi_y \in (0, \infty)$ such that $\pi_i^* (\varphi_y) = y$.

Define: $\Psi_i \equiv V_i^{in} - V_i^{out}$. Assume for the moment that $\Psi_i < \frac{\mu_i \kappa_i + c}{\delta}$. Then: $-c < \mu_i \kappa_i - \delta \Psi_i < \kappa_i - \delta \Psi_i$.

A firm that is already in the market has incentive to stay in the market (rather than exit) iff:

$$\mu_i \kappa_i + \delta V_i^{out} \leq \pi_i^* (\varphi) + \delta V_i^{in} \quad \Leftrightarrow \quad \mu_i \kappa_i - \delta \Psi_i \leq \pi_i^* (\varphi)$$

Define: $x_i \equiv \pi_i^{*-1} (\mu_i \kappa_i - \delta \Psi_i)$. This is the type of firm that is in the market and indifferent about whether to remain.

A firm that is not already in the market has incentive to enter the market iff:

$$\delta V_i^{out} \leq \pi_i^* (\varphi) - \kappa_i + \delta V_i^{in} \quad \Leftrightarrow \quad \kappa_i - \delta \Psi_i \leq \pi_i^* (\varphi)$$

Define: $y_i \equiv \pi_i^{*-1} (\kappa_i - \delta \Psi_i)$. This is the type of firm that is out of the market and indifferent about whether to enter.

Because $\pi_i^* (\varphi)$ is an increasing function and $\mu_i \kappa_i - \delta \Psi_i < \kappa_i - \delta \Psi_i$, it must be true that: $x_i < y_i$.

1.4 Weighted Average Productivity

To simplify calculations, define the weighted average productivity of firms as:

$$\widetilde{\varphi}_x = \widetilde{\varphi} (x) \equiv \left[\frac{1}{1 - G(x)} \int_x^\infty \varphi^{\sigma - 1} dG (\varphi) \right]^{\frac{1}{\sigma - 1}} \quad \Leftrightarrow \quad \widetilde{\varphi}_x^{\sigma - 1} [1 - G(x)] = \int_x^\infty \varphi^{\sigma - 1} dG (\varphi)$$

Note that this implies that the weighted average profitability of firms, given cutpoint x_i, is:

$$\begin{aligned}
\int_{x_i}^\infty \pi_i^* (\varphi) \, dG (\varphi) &= \int_{x_i}^\infty \left(\phi_i \varphi^{\sigma - 1} - c \right) dG (\varphi) = \phi_i \int_{x_i}^\infty \varphi^{\sigma - 1} dG (\varphi) - c [1 - G(x_i)] \\
&= \phi_i \widetilde{\varphi}_{x_i}^{\sigma - 1} [1 - G(x_i)] - c [1 - G(x_i)] = \left(\phi_i \widetilde{\varphi}_{x_i}^{\sigma - 1} - c \right) [1 - G(x_i)] \\
&= \pi_i^* (\widetilde{\varphi}_{x_i}) [1 - G(x_i)]
\end{aligned}$$

1.5 Continuation Values and Free Entry

We continue to suppress industry subscripts to simplify notation.

The continuation value for a firm is its expected utility prior to learning its type, φ, in a given period. For $i \in \{d, f\}$, the continuation value for a firm that is already in the market is:

$$\begin{aligned}
V_i^{in} &= \int_0^{x_i} \left(\mu_i \kappa_i + \delta V_i^{out} \right) dG (\varphi) + \int_{x_i}^\infty \left[\pi_i^* (\varphi) + \delta V_i^{in} \right] dG (\varphi) \\
&= \left(\mu_i \kappa_i + \delta V_i^{out} \right) G(x_i) + \delta V_i^{in} [1 - G(x_i)] + \int_{x_i}^\infty \pi_i^* (\varphi) \, dG (\varphi) \\
&= \frac{\left(\mu_i \kappa_i + \delta V_i^{out} \right) G(x_i) + \pi^* (\widetilde{\varphi}_{x_i}) [1 - G(x_i)]}{1 - \delta [1 - G(x_i)]}
\end{aligned}$$

For $i \in \{d, f\}$, the continuation value for a firm that is not already in the market is:

$$\begin{aligned} V_i^{out} &= \int_0^{y_i} \delta V_i^{out} dG(\varphi) + \int_{y_i}^{\infty} \left[\pi_i^*(\varphi) - \kappa_i + \delta V_i^{in}\right] dG(\varphi) \\ &= \delta V_i^{out} G(y_i) + \left(\delta V_i^{in} - \kappa_i\right)\left[1 - G(y_i)\right] + \int_{y_i}^{\infty} \pi_i^*(\varphi)\, dG(\varphi) \\ &= \frac{\left[\delta V_i^{in} - \kappa_i + \pi_i^*(\widetilde{\varphi}_{y_i})\right]\left[1 - G(y_i)\right]}{1 - \delta G(y_i)} \end{aligned}$$

Isolating the continuation values and performing substitutions yields the following:

$$V_i^{in} = \frac{\gamma_i^{in}\kappa_i + \left[1 - G(x_i)\right]\left[1 - \delta G(y_i)\right]\pi^*(\widetilde{\varphi}_{x_i}) + \delta G(x_i)\left[1 - G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{y_i})}{\theta_i} \quad (1)$$

$$V_i^{out} = \frac{\gamma_i^{out}\kappa_i + \delta\left[1 - G(x_i)\right]\left[1 - G(y_i)\right]\pi^*(\widetilde{\varphi}_{x_i}) + \left[1 - \delta + \delta G(x_i)\right]\left[1 - G(y_i)\right]\pi_i^*(\widetilde{\varphi}_{y_i})}{\theta_i} \quad (2)$$

where:

$$\begin{aligned} \theta_i &\equiv (1 - \delta)\left[1 + \delta G(x_i) - \delta G(y_i)\right] \\ \gamma_i^{in} &\equiv G(x_i)\left\{\mu_i\left[1 - \delta G(y_i)\right] - \delta\left[1 - G(y_i)\right]\right\} \\ \gamma_i^{out} &\equiv \left\{\delta\mu_i G(x_i) - \left[1 - \delta + \delta G(x_i)\right]\right\}\left[1 - G(y_i)\right] \end{aligned}$$

Recall that β_{ij}^{in} is the informational cost for an (ij)-firm that is currently "in" the market to learn its type in the given period, and β_{ij}^{out} is the informational cost for an (ij)-firm that is currently "out" of the market to learn its type in the given period. Free entry requires that for every $i \in \{d, f\}$ and every industry j: $V_{ij}^{in} = \beta_{ij}^{in}$ and $V_{ij}^{out} = \beta_{ij}^{out}$.[1]

So it must be true that $\Psi_{ij} = \beta_{ij}^{in} - \beta_{ij}^{out}$ for $i \in \{d, f\}$ and every j.

A necessary condition for our equilibrium is therefore that $\beta_{ij}^{in} - \beta_{ij}^{out} < \frac{\mu_{ij}\kappa_{ij} + c}{\delta}$.

1.6 Zero Profit Conditions

We continue to suppress industry subscripts to simplify notation.

Because "in" firms are indifferent about whether to exit at cutpoint x_i, and "out" firms are indifferent about whether to enter at cutpoint y_i, we have four zero profit conditions for each industry:

$$\begin{aligned} \pi_i^*(x_i) &= \mu_i\kappa_i - \delta\Psi_i \\ \pi_i^*(y_i) &= \kappa_i - \delta\Psi_i \qquad \text{for: } i \in \{d, f\} \end{aligned}$$

Define $\alpha_i \equiv c - \delta\Psi_i$. The zero profit conditions imply that for $i \in \{d, f\}$:

$$\begin{aligned} r_i^*(x_i) &= \sigma(\mu_i\kappa_i + \alpha_i) \\ r_i^*(y_i) &= \sigma(\kappa_i + \alpha_i) \end{aligned}$$

Note that:

$$\frac{r_d^*(x_d)}{r_d^*(y_d)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^{\sigma}}{\left(\frac{y_d}{\rho}\right)^{\sigma-1} QP^{\sigma}} = \left(\frac{x_d}{y_d}\right)^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\kappa_d + \alpha_d)} \quad \Leftrightarrow \quad y_d = x_d\left(\frac{\kappa_d + \alpha_d}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}$$

[1] This is equivalent to the free entry condition condition in equation (11) of **?**, as our continuation values take into account uncertainty about movement both in and out of the market over time.

And:

$$\frac{r_d^*(x_d)}{r_f^*(x_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left[\frac{x_f}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma} = \left[\frac{x_d(1+\tau)}{x_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\mu_f\kappa_f + \alpha_f)} \quad \Leftrightarrow \quad x_f = x_d(1+\tau)\left(\frac{\mu_f\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}$$

And:

$$\frac{r_d^*(x_d)}{r_f^*(y_f)} = \frac{\left(\frac{x_d}{\rho}\right)^{\sigma-1} QP^\sigma}{\left[\frac{y_f}{\rho(1+\tau)}\right]^{\sigma-1} QP^\sigma} = \left[\frac{x_d(1+\tau)}{y_f}\right]^{\sigma-1} = \frac{\sigma(\mu_d\kappa_d + \alpha_d)}{\sigma(\kappa_f + \alpha_f)} \quad \Leftrightarrow \quad y_f = x_d(1+\tau)\left(\frac{\kappa_f + \alpha_f}{\mu_d\kappa_d + \alpha_d}\right)^{\frac{1}{\sigma-1}}$$

So for each industry $j = 1, \ldots, J$, the cutpoints (y_{dj}, x_{fj}, y_{fj}) can all be expressed in terms of variable x_{dj}:

$$\begin{aligned}
y_{dj}(x_{dj}) &= \eta_{1j} x_{dj} \qquad \text{where: } \eta_{1j} \equiv \left(\frac{\kappa_{dj} + \alpha_{dj}}{\mu_{dj}\kappa_{dj} + \alpha_{dj}}\right)^{\frac{1}{\sigma-1}} \\
x_{fj}(x_{dj}) &= \eta_{2j}(1+\tau_j)\, x_{dj} \qquad \text{where: } \eta_{2j} \equiv \left(\frac{\mu_{fj}\kappa_{fj} + \alpha_{fj}}{\mu_{dj}\kappa_{dj} + \alpha_{dj}}\right)^{\frac{1}{\sigma-1}} \\
y_{fj}(x_{dj}) &= \eta_{3j}(1+\tau_j)\, x_{dj} \qquad \text{where: } \eta_{3j} \equiv \left(\frac{\kappa_{fj} + \alpha_{fj}}{\mu_{dj}\kappa_{dj} + \alpha_{dj}}\right)^{\frac{1}{\sigma-1}}
\end{aligned}$$

1.7 Firm Masses

For $i \in \{d, f\}$ and $j = 1, \ldots, J$, the total mass of (ij)-firms is:

$$M_{ij} = M_{ij}^{in} + M_{ij}^{out}$$

Stationarity requires the following:

$$M_{ij}^{in} = [1 - G(x_{ij})]\, M_{ij}^{in} + [1 - G(y_{ij})]\, M_{ij}^{out} \quad \Leftrightarrow \quad M_{ij}^{in} = \left[\frac{1 - G(y_{ij})}{G(x_{ij})}\right] M_{ij}^{out}$$

1.8 Labor Market Clearing

We continue to suppress industry subscripts to simplify notation. Note that the distribution of new producers (firms that were "out", but then entered) differs from the distribution of old producers (firms that were "in" and then stayed). Namely:

$$h_i^{old} = \begin{cases} \frac{g(\varphi)}{1-G(x_i)} & \text{if } x_i \le \varphi \\ 0 & \text{if } \varphi < x_i \end{cases} \quad \text{and} \quad h_i^{new} = \begin{cases} \frac{g(\varphi)}{1-G(y_i)} & \text{if } y_i \le \varphi \\ 0 & \text{if } \varphi < y_i \end{cases}$$

Aggregate profit for i firms in a specific period t is:

$$\begin{aligned}
\Pi_i^t &= \underbrace{[1 - G(x_i)]\, M_i^{in}}_{\text{mass of old firms (that stay)}} \underbrace{\int_{x_i}^{\infty} \pi_i^*(\varphi)\, h_i^{old}(\varphi)\, d\varphi}_{\text{old firm profits}} + \underbrace{[1 - G(y_i)]\, M_i^{out}}_{\text{mass of new firms}} \underbrace{\int_{y_i}^{\infty} \pi_i^*(\varphi)\, h_i^{new}(\varphi)\, d\varphi}_{\text{new firm profits}} \\
&= M_i^{in} \int_{x_i}^{\infty} \pi_i^*(\varphi)\, dG(\varphi) + M_i^{out} \int_{y_i}^{\infty} \pi_i^*(\varphi)\, dG(\varphi) \\
&= M_i^{in} \pi_i^*(\widetilde{\varphi}_{x_i})\, [1 - G(x_i)] + M_i^{out} \pi_i^*(\widetilde{\varphi}_{y_i})\, [1 - G(y_i)]
\end{aligned}$$

The present value of aggregate profit for i firms over time is therefore:

$$\Pi_i = M_i^{in}\pi_i^*(\widetilde{\varphi}_{xi})[1-G(x_i)] + M_i^{out}\pi_i^*(\widetilde{\varphi}_{yi})[1-G(y_i)] + \delta M_i^{in}V_i^{in} + \delta M_i^{out}V_i^{out} \tag{3}$$

Recall that given the free entry conditions and the definition of continuation values in equations (1) and (2):

$$\begin{aligned}
\beta_i^{in} &= V_i^{in} = \frac{\gamma_i^{in}\kappa_i + [1-G(x_i)][1-\delta G(y_i)]\pi^*(\widetilde{\varphi}_{x_i}) + \delta G(x_i)[1-G(y_i)]\pi_i^*(\widetilde{\varphi}_{y_i})}{\theta_i} \\
\beta_i^{out} &= V_i^{out} = \frac{\gamma_i^{out}\kappa_i + \delta[1-G(x_i)][1-G(y_i)]\pi^*(\widetilde{\varphi}_{x_i}) + [1-\delta+\delta G(x_i)][1-G(y_i)]\pi_i^*(\widetilde{\varphi}_{y_i})}{\theta_i}
\end{aligned}$$

By manipulating these conditions, we can first isolate $\pi^*(\widetilde{\varphi}_{x_i})$ and $\pi^*(\widetilde{\varphi}_{y_i})$, and then use substitutions to show that:

$$\begin{aligned}
\pi^*(\widetilde{\varphi}_{xi})[1-G(x_i)] &= [1-\delta+\delta G(x_i)]\beta_i^{in} - \delta G(x_i)\beta_i^{out} - G(x_i)\mu_i\kappa_i \\
\pi_i^*(\widetilde{\varphi}_{yi})[1-G(y_i)] &= [1-\delta G(y_i)]\beta_i^{out} - \delta[1-G(y_i)]\beta_i^{in} + [1-G(y_i)]\kappa_i
\end{aligned}$$

Substitution of these terms and the free entry conditions into equation (3) yields:

$$\begin{aligned}
\Pi_i &= M_i^{in}\left\{[1-\delta+\delta G(x_i)]\beta_i^{in} - \delta G(x_i)\beta_i^{out} - G(x_i)\mu_i\kappa_i\right\} \\
&\quad + M_i^{out}\left\{[1-\delta G(y_i)]\beta_i^{out} - \delta[1-G(y_i)]\beta_i^{in} + [1-G(y_i)]\kappa_i\right\} + \delta M_i^{in}\beta_i^{in} + \delta M_i^{out}\beta_i^{out} \\
&= \left\{M_i^{in}[1+\delta G(x_i)] - M_i^{out}\delta[1-G(y_i)]\right\}\beta_i^{in} \\
&\quad + \left\{M_i^{out}[1+\delta-\delta G(y_i)] - M_i^{in}\delta G(x_i)\right\}\beta_i^{out} + [1-G(y_i)]M_i^{out}\kappa_i - G(x_i)M_i^{in}\mu_i\kappa_i
\end{aligned}$$

Recall from above that stationarity requires that: $M_i^{in} = \left[\frac{1-G(y_i)}{G(x_i)}\right]M_i^{out}$. This is turn implies that: $M_i^{out} = \left[\frac{G(x_i)}{1-G(y_i)}\right]M_i^{in}$. So:

$$\Pi_i = \underbrace{M_i^{in}\beta_i^{in} + M_i^{out}\beta_i^{out}}_{L_{ti}} + \underbrace{[1-G(y_i)]M_i^{out}\kappa_i}_{L_{si}} - \underbrace{G(x_i)M_i^{in}\mu_i\kappa_i}_{L_{ri}}$$

where:

- L_{ti} is labor spent on learning each firm's type
- L_{si} is labor that "out" firms spend on setting up new production when they enter the market
- L_{ri} is labor that is recovered when "in" firms decide to exit the market

Total revenue by i firms is the total profits, plus the total production costs, L_{pi}. Then for $i \in \{d, f\}$:

$$R_i = \Pi_i + L_{pi} = L_{pi} + L_{ti} + L_{si} - L_{ri} = L_i$$

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We now reintroduce the industry subscript to focus on labor allocation at the industry-level.
Note that for each industry j: $R_j = R_{dj} + R_{fj} = L_{dj} + L_{fj} = L_j$.
Also, given the equilibrium demand function, the revenue generate in industry j (across all types of firms) is:

$$\begin{aligned}
R_j &= \int\limits_{v\in V_j} p_j(v)\,q_j(v)\,dv = \int\limits_{v\in V_j} p_j(v)^{1-\sigma} Q_j P_j^{\sigma}\,dv \\
&= Q_j P_j^{\sigma}\int\limits_{v\in V_j} p_j(v)^{1-\sigma}\,dv = Q_j P_j^{\sigma} P_j^{1-\sigma} = P_j Q_j
\end{aligned}$$

So by going back to our original Lagrangian, we can see that:

$$\mathcal{L} = \left(1-\sum_{j=1} w_j\right)\log v_0 + \sum_{j=1} w_j \log Q_j + \lambda\left[R-\sum_{j=0} P_j Q_j\right]$$

Then for $j \in \{1,\ldots,J\}$, the first-order condition at the industry-level becomes:

$$\frac{\partial\mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \lambda P_j = 0 \quad \Leftrightarrow \quad \lambda = \frac{w_j}{P_j Q_j}$$

And:

$$1 = \sum_{j=0} w_j = \lambda \sum_{j=0} P_j Q_j = \lambda \sum_{j=0} R_j = \lambda R = \lambda L \quad \Leftrightarrow \quad \lambda = \frac{1}{L}$$

So:

$$\frac{\partial\mathcal{L}}{\partial Q_j} = \frac{w_j}{Q_j} - \frac{P_j}{L} = 0 \quad \Leftrightarrow \quad L w_j = P_j Q_j = R_j = L_j$$

So the labor market across all industries clears.

1.9 Equilibrium Characterization under the Pareto Distribution

Suppose that types are chosen according to the Pareto distribution, iid over time and players, with domain $\varphi \sim [b,\infty)$ for small $b>0$ and shape parameter $a>\sigma$. The Pareto distribution has the following attributes:

$$\text{Density function:} \quad g(\varphi) = \frac{ab^a}{\varphi^{a+1}} \quad \text{for: } \varphi \geq b$$

$$\text{Distribution function:} \quad G(y) = \Pr(\varphi \leq y) = 1-\left(\frac{b}{y}\right)^a \quad \text{for: } y \geq b$$

Define $z \equiv a-\sigma+1$. Note that under the Pareto distribution, for $x \geq b$:

$$\begin{aligned}
\int_x^\infty \varphi^{\sigma-1} dG(\varphi) &= \int_x^\infty \varphi^{\sigma-1}\left(\frac{ab^a}{\varphi^{a+1}}\right) d\varphi = ab^a \int_x^\infty \varphi^{\sigma-a-2} d\varphi \\
&= ab^a \lim_{t\to\infty} \int_x^t \varphi^{\sigma-a-2} d\varphi = ab^a \lim_{t\to\infty}\left[\frac{\varphi^{\sigma-a-1}}{\sigma-a-1}\right]_x^t = -\frac{ab^a}{z}\lim_{t\to\infty}\left[\frac{1}{t^z}-\frac{1}{x^z}\right] \\
&= \frac{ab^a}{z}\left(\frac{1}{x^z}\right) = \frac{ab^a}{z}x^{-z}
\end{aligned}$$

Recall that for each industry $j=1,\ldots,J$, each set of cutpoints for equilibrium behavior can be expressed in terms of the cutpoint x_{dj}. Equilibrium behavior in industry j is therefore defined by the function:

$$\pi^*_{dj}(x_{dj}) = \phi_{dj} x_{dj}^{\sigma-1} - c = \mu_{dj}\kappa_{dj} - \delta\Psi_{dj}$$

where:

$$\phi_{dj} = \frac{Q_j P_j^\sigma}{\sigma\rho^{\sigma-1}} = \frac{R_j P_j^{\sigma-1}}{\sigma\rho^{\sigma-1}} = \frac{L w_j P_j^{\sigma-1}}{\sigma\rho^{\sigma-1}}$$

The equilibrium price index is:

$$
\begin{aligned}
P_j(x_{dj}) &= \rho\left[\int_{x_{dj}}^{\infty}\varphi^{\sigma-1}dG(\varphi)+\int_{\eta_{1j}x_{dj}}^{\infty}\varphi^{\sigma-1}dG(\varphi)\right.\\
&\qquad\left.+(1+\tau_j)^{1-\sigma}\left(\int_{\eta_{2j}(1+\tau_j)x_{dj}}^{\infty}\varphi^{\sigma-1}dG(\varphi)+\int_{\eta_{3j}(1+\tau_j)x_{dj}}^{\infty}\varphi^{\sigma-1}dG(\varphi)\right)\right]^{\frac{1}{1-\sigma}}\\
&= \rho\left\{\frac{ab^a}{z}x_{dj}^{-z}+\frac{ab^a}{z}(\eta_{1j}x_{dj})^{-z}+(1+\tau_j)^{1-\sigma}\left[\frac{ab^a}{z}(\eta_{2j}(1+\tau_j)x_{dj})^{-z}+\frac{ab^a}{z}(\eta_{3j}(1+\tau_j)x_{dj})^{-z}\right]\right\}^{\frac{1}{1-\sigma}}\\
&= \rho\left(\frac{zx_{dj}^{z}}{ab^a\Phi_j}\right)^{\frac{1}{\sigma-1}} \qquad \text{where: } \Phi_j\equiv 1+\eta_{1j}^{-z}+(1+\tau_j)^{-a}\left(\eta_{2j}^{-z}+\eta_{3j}^{-z}\right)
\end{aligned}
$$

And:

$$
P_j(x_{dj})^{\sigma-1}=\frac{\rho^{\sigma-1}zx_{dj}^{z}}{ab^a\Phi_j}
$$

So the best response function is:

$$
\begin{aligned}
\Rightarrow \pi_{dj}^{*}(x_{dj}) &= \left(\frac{Lw_jP_j^{\sigma-1}}{\sigma\rho^{\sigma-1}}\right)x_{dj}^{\sigma-1}-c=\mu_{dj}\kappa_{dj}-\delta\Psi_{dj}\\
&\Leftrightarrow \left(\frac{Lw_j}{\sigma\rho^{\sigma-1}}\right)\left(\frac{\rho^{\sigma-1}zx_{dj}^{z}}{ab^a\Phi_j}\right)x_{dj}^{\sigma-1}=\mu_{dj}\kappa_{dj}+\alpha_{dj}\\
&\Leftrightarrow x_{dj}^{a}=\left(\frac{ab^a\sigma}{zLw_j}\right)(\mu_{dj}\kappa_{dj}+\alpha_{dj})\Phi_j\\
&\Leftrightarrow x_{dj}^{*}=\psi_j\Phi_j^{\frac{1}{a}} \qquad \text{where: } \psi_j\equiv\left[\frac{ab^a\sigma}{zLw_j}(\mu_{dj}\kappa_{dj}+\alpha_{dj})\right]^{\frac{1}{a}}
\end{aligned}
$$

]

QED.

Proof of Proposition 2.

The average weighted productivity of (ij)-firms in any given period is:

$$
\widetilde{\varphi}_{ij} = \left\{\frac{1}{1-G(x_{ij})+1-G(y_{ij})}\left[\int_{x_{ij}}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi+\int_{y_{ij}}^{\infty}\varphi^{\sigma-1}g(\varphi)\,d\varphi\right]\right\}^{\frac{1}{\sigma-1}}
$$

Under the Pareto distribution:

$$
\widetilde{\varphi}_{ij}=\left\{\frac{1}{\left(\frac{b}{x_{ij}}\right)^{a}+\left(\frac{b}{y_{ij}}\right)^{a}}\left[\frac{ab^a}{z}x_{ij}^{-z}+\frac{ab^a}{z}y_{ij}^{-z}\right]\right\}^{\frac{1}{\sigma-1}}=\left[\frac{a\left(x_{ij}^{-z}+y_{ij}^{-z}\right)}{z\left(x_{ij}^{-a}+y_{ij}^{-a}\right)}\right]^{\frac{1}{\sigma-1}}
$$

We now suppress the industry subscript to simplify notation. For foreign firms:

$$\begin{aligned}
\widetilde{\varphi}_f &= \left[\frac{a\left(x_f^{-z}+y_f^{-z}\right)}{z\left(x_f^{-a}+y_f^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left([\eta_2(1+\tau)x_d]^{-z}+[\eta_3(1+\tau)x_d]^{-z}\right)}{z\left([\eta_2(1+\tau)x_d]^{-a}+[\eta_3(1+\tau)x_d]^{-a}\right)}\right]^{\frac{1}{\sigma-1}} \\
&= \left[\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)(1+\tau)^{-z}x_d^{-z}}{z\left(\eta_2^{-a}+\eta_3^{-a}\right)(1+\tau)^{-a}x_d^{-a}}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)(1+\tau)^{\sigma-1}x_d^{\sigma-1}}{z\left(\eta_2^{-a}+\eta_3^{-a}\right)}\right]^{\frac{1}{\sigma-1}} \\
&= \nu(1+\tau)x_d H \quad \text{where: } \nu \equiv \left(\frac{a}{z}\right)^{\frac{1}{\sigma-1}} \text{ and } H \equiv \left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial\widetilde{\varphi}_f}{\partial\tau} &= \nu H\left[(1+\tau)\frac{\partial x_d}{\partial\tau}+x_d\right] = \frac{\nu H x_d}{a\Phi}\left[(1+\tau)\frac{\partial\Phi}{\partial\tau}+a\Phi\right] \\
&= \frac{\nu H x_d}{\Phi}\left[\Phi-(1+\tau)^{-a}\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{\nu\left(1+\eta_1^{-z}\right)H x_d}{\Phi} > 0
\end{aligned}$$

For domestic firms:

$$\begin{aligned}
\widetilde{\varphi}_d &= \left[\frac{a\left(x_d^{-z}+y_d^{-z}\right)}{z\left(x_d^{-a}+y_d^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(x_d^{-z}+(\eta_1 x_d)^{-z}\right)}{z\left(x_d^{-a}+(\eta_1 x_d)^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(1+\eta_1^{-z}\right)x_d^{-z}}{z\left(1+\eta_1^{-a}\right)x_d^{-a}}\right]^{\frac{1}{\sigma-1}} \\
&= \left[\frac{a\left(1+\eta_1^{-z}\right)x_d^{\sigma-1}}{z\left(1+\eta_1^{-a}\right)}\right]^{\frac{1}{\sigma-1}} = \left[\frac{a\left(1+\eta_1^{-z}\right)}{z\left(1+\eta_1^{-a}\right)}\right]^{\frac{1}{\sigma-1}} x_d
\end{aligned}$$

$$\frac{\partial\widetilde{\varphi}_d}{\partial\tau} = \left[\frac{a\left(1+\eta_1^{-z}\right)}{z\left(1+\eta_1^{-a}\right)}\right]^{\frac{1}{\sigma-1}}\frac{\partial x_d}{\partial\tau} < 0$$

QED.

Proof of Proposition 3.

Firm-level production by foreign firms in industry j is:

$$q_{fj}(\varphi) = \left[\frac{\varphi}{\rho(1+\tau_j)}\right]^{\sigma} Q_j P_j^{\sigma}$$

So aggregate production by foreign firms in industry j is:

$$\begin{aligned}
Q_{fj}(\tau_j) &= \int_{x_{fj}}^{\infty} q_{fj}(\varphi) g(\varphi) d\varphi + \int_{y_{fj}}^{\infty} q_{fj}(\varphi) g(\varphi) d\varphi \\
&= \frac{Q_j P_j^{\sigma}}{\rho^{\sigma}(1+\tau_j)^{\sigma}} \left[\int_{x_{fj}}^{\infty} \varphi^{\sigma} g(\varphi) d\varphi + \int_{y_{fj}}^{\infty} \varphi^{\sigma} g(\varphi) d\varphi \right] \\
&= \frac{ab^a R_j P_j^{\sigma-1}}{(z-1)\rho^{\sigma}(1+\tau_j)^{\sigma}} \left(x_{fj}^{1-z} + y_{fj}^{1-z} \right) \\
&= \frac{ab^a L w_j}{(z-1)\rho^{\sigma}(1+\tau_j)^{\sigma}} \left(\frac{\rho^{\sigma-1} z x_{dj}^{z}}{ab^a \Phi_j} \right) \left\{ \left[\eta_{2j}(1+\tau_j) x_{dj}\right]^{1-z} + \left[\eta_{3j}(1+\tau_j) x_{dj}\right]^{1-z} \right\} \\
&= \frac{z L w_j}{(z-1)\rho(1+\tau_j)^{\sigma}} \left(\frac{x_{dj}^{z}}{\Phi_j} \right) \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) (1+\tau_j)^{1-z} x_{dj}^{1-z} \\
&= \frac{z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) x_{dj}}{(z-1)\rho(1+\tau_j)^{a} \Phi_j} = \frac{z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) \psi_j \Phi_j^{\frac{1-a}{a}}}{(z-1)\rho(1+\tau_j)^{a}}
\end{aligned}$$

Note that:

$$\begin{aligned}
\frac{\partial Q_{fj}(\tau_j)}{\partial \tau_j} &= \frac{z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) \psi_j}{(z-1)\rho} \left[\frac{(1+\tau_j)^a \left(\frac{1-a}{a}\right) \Phi_j^{\frac{1-2a}{a}} \left(\frac{\partial \Phi_j}{\partial \tau_j} \right) - a \Phi_j^{\frac{1-a}{a}} (1+\tau_j)^{a-1}}{(1+\tau_j)^{2a}} \right] \\
&= \frac{z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) \psi_j}{(z-1)\rho} \left[\frac{(1+\tau_j)^a \left(\frac{1-a}{a}\right) \Phi_j^{\frac{1-2a}{a}} \left(-a(1+\tau_j)^{-a-1} \left(\eta_{2j}^{-z} + \eta_{3j}^{-z} \right) \right) - a \Phi_j^{\frac{1-a}{a}} (1+\tau_j)^{a-1}}{(1+\tau_j)^{2a}} \right] \\
&= \frac{-z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) \psi_j \Phi_j^{\frac{1-2a}{a}}}{(z-1)\rho} \left[\frac{a \Phi_j (1+\tau_j)^a - (a-1) \left(\eta_{2j}^{-z} + \eta_{3j}^{-z} \right)}{(1+\tau_j)^{2a+1}} \right] \\
&= \frac{-z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) x_{dj}}{(z-1)\rho \Phi_j^2} \left[\frac{a \left[1 + \eta_{1j}^{-z} + (1+\tau_j)^{-a} \left(\eta_{2j}^{-z} + \eta_{3j}^{-z} \right) \right] (1+\tau_j)^a - (a-1) \left(\eta_{2j}^{-z} + \eta_{3j}^{-z} \right)}{(1+\tau_j)^{2a+1}} \right] \\
&= \frac{-z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z} \right) x_{dj}}{(z-1)\rho \Phi_j^2} \left[\frac{a (1+\tau_j)^a \left(1 + \eta_{1j}^{-z} \right) + \left(\eta_{2j}^{-z} + \eta_{3j}^{-z} \right)}{(1+\tau_j)^{2a+1}} \right]
\end{aligned}$$

The government's objective function is:

$$U = \sum_{j=0}^{J} \tau_j Q_{fj}(\tau_j)$$

The first order condition for takings on industry $j = 1, \ldots, J$ is thus:

$$\begin{aligned}
\frac{\partial U}{\partial \tau_j} &= \tau_j \frac{\partial Q_{fj}(\tau_j)}{\partial \tau_j} + Q_{fj}(\tau_j) \\
&= \frac{-\tau_j z L w_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z}\right) x_{dj}}{(z-1)\rho \Phi_j^2} \left[\frac{a(1+\tau_j)^a \left(1+\eta_{1j}^{-z}\right) + \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right)}{(1+\tau_j)^{2a+1}}\right] + \frac{zLw_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z}\right) x_{dj}}{(z-1)\rho (1+\tau_j)^a \Phi_j} \\
&= \frac{zLw_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z}\right) x_{dj}}{(z-1)\rho (1+\tau_j)^{2a+1} \Phi_j^2} \left\{(1+\tau_j)^{a+1} \Phi_j - \tau_j \left[a(1+\tau_j)^a \left(1+\eta_{1j}^{-z}\right) + \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right)\right]\right\} \\
&= \frac{zLw_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z}\right) x_{dj}}{(z-1)\rho (1+\tau_j)^{2a+1} \Phi_j^2} \left\{(1+\tau_j)^{a+1} \left[1 + \eta_{1j}^{-z} + (1+\tau_j)^{-a} \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right)\right]\right. \\
&\qquad \left. - \tau_j \left[a(1+\tau_j)^a \left(1+\eta_{1j}^{-z}\right) + \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right)\right]\right\} \\
&= \frac{zLw_j \left(\eta_{2j}^{1-z} + \eta_{3j}^{1-z}\right) x_{dj}}{(z-1)\rho (1+\tau_j)^{a+1} \Phi_j^2} \left[(1+\tau_j - a\tau_j)\left(1+\eta_{1j}^{-z}\right) + (1+\tau_j)^{-a} \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right)\right]
\end{aligned}$$

Note that:

$$\frac{\partial U}{\partial \tau_j} \geq 0 \Leftrightarrow 0 \leq (1+\tau_j - a\tau_j)\left(1+\eta_{1j}^{-z}\right) + (1+\tau_j)^{-a} \left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right) \equiv \Gamma(\tau_j)$$

Additionally, note that:

$$\Gamma(\tau_j = 0) = 1 + \eta_{1j}^{-z} + \eta_{2j}^{-z} + \eta_{3j}^{-z} > 0$$

$$\frac{\partial \Gamma(\tau_j)}{\partial \tau_j} = -(a-1)\left(1+\eta_{1j}^{-z}\right) - a(1+\tau_j)^{-a-1}\left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right) < 0$$

$$\lim_{\tau_j \to \infty} \Gamma(\tau_j) = \lim_{\tau_j \to \infty} \left\{[1 - \tau_j(a-1)]\left(1+\eta_1^{-z}\right) + \frac{\eta_2^{-z} + \eta_3^{-z}}{(1+\tau_j)^a}\right\} = -\infty < 0$$

So by the intermediate value theorem, there is a unique optimal takings rate. Furthermore:

$$\frac{\partial U}{\partial \tau_j} = 0 \Leftrightarrow \Gamma(\tau_j) = 0$$

So the political equilibrium for a given industry is characterized by the implicitly defined variable τ_j^*, where τ_j^* solves:

$$\Gamma_j\left(\tau_j^*\right) = \left(1 + \tau_j^* - \tau_j^* a\right)\left(1+\eta_{1j}^{-z}\right) + \left(1+\tau_j^*\right)^{-a}\left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right) = 0$$

This solution in turn determines the equilibrium cutpoints, $\left(x_{dj}^*, y_{dj}^*, x_{fj}^*, y_{fj}^*\right)$.

QED.

Proof of Proposition 4.

Recall from the Proof of Proposition 3 that the equilibrium taking rate for industry $j = 1, \ldots, J$, τ_j, solves:

$$\Gamma_j\left(\tau_j^*\right) = \left(1 + \tau_j^* - \tau_j^* a\right)\left(1+\eta_{1j}^{-z}\right) + \left(1+\tau_j^*\right)^{-a}\left(\eta_{2j}^{-z} + \eta_{3j}^{-z}\right) = 0$$

So by the implicit function theorem, for any exogenous variable y_j:

$$\frac{\partial \tau_j^*}{\partial y_j} = \frac{-\Gamma_{y_j}}{\Gamma_{\tau_j^*}}$$

Note that:

$$\begin{aligned}
\frac{\partial\eta_{2j}}{\partial\kappa_{fj}} &= \frac{1}{\sigma-1}\left(\frac{\mu_{fj}\kappa_{fj}+\alpha_{fj}}{\mu_{dj}\kappa_{dj}+\alpha_{dj}}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{\mu_{fj}}{\mu_{dj}\kappa_{dj}+\alpha_{dj}}\right) = \frac{\mu_{fj}\eta_{2j}}{(\sigma-1)\left(\mu_{fj}\kappa_{fj}+\alpha_{fj}\right)} > 0 \\
\frac{\partial\eta_{3j}}{\partial\kappa_{fj}} &= \frac{1}{\sigma-1}\left(\frac{\kappa_{fj}+\alpha_{fj}}{\mu_{dj}\kappa_{dj}+\alpha_{dj}}\right)^{\frac{1}{\sigma-1}-1}\left(\frac{1}{\mu_{dj}\kappa_{dj}+\alpha_{dj}}\right) = \frac{\eta_{3j}}{(\sigma-1)\left(\kappa_{fj}+\alpha_{fj}\right)} > 0
\end{aligned}$$

And:

$$\begin{aligned}
\Gamma_{\tau_j^*} &= -\left[(a-1)\left(1+\eta_{1j}^{-z}\right) + a\left(1+\tau_j^*\right)^{-a-1}\left(\eta_{2j}^{-z}+\eta_{3j}^{-z}\right)\right] < 0 \\
\Gamma_{\kappa_f} &= -z\left(1+\tau_j^*\right)^{-a}\left[\eta_{2j}^{-z-1}\left(\frac{\partial\eta_{2j}}{\partial\kappa_{fj}}\right) + \eta_{3j}^{-z-1}\left(\frac{\partial\eta_{3j}}{\partial\kappa_{fj}}\right)\right] < 0
\end{aligned}$$

So:

$$\frac{\partial\tau_j^*}{\partial\kappa_{fj}} = \frac{-\Gamma_{\kappa_f}}{\Gamma_{\tau^*}} < 0$$

QED.

Proof of Proposition 5.

We suppress industry notation. As shown in the Proof for Proposition 2, the average weighted productivity for foreign firms in any given period is:

$$\widetilde{\varphi}_f = \nu\left(1+\tau\right)x_d H \quad \text{where: } \nu \equiv \left(\frac{a}{z}\right)^{\frac{1}{\sigma-1}} \text{ and } H \equiv \left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}}$$

Define:

$$\widehat{\eta} \equiv \eta_3 = \lim_{\mu_f\to 1}\eta_2 = \left(\frac{\kappa_{fj}+\alpha_{fj}}{\mu_{dj}\kappa_{dj}+\alpha_{dj}}\right)^{\frac{1}{\sigma-1}}$$

Define:

$$\begin{aligned}
\phi(-z) \equiv \frac{\partial\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\partial\kappa_f} &= -z\eta_2^{-z-1}\frac{\partial\eta_2}{\partial\kappa_f} - z\eta_3^{-z-1}\frac{\partial\eta_3}{\partial\kappa_f} = -\left(\frac{z}{\sigma-1}\right)\left[\frac{\mu_f\eta_2^{-z}}{\mu_f\kappa_f+\alpha_f} + \frac{\eta_3^{-z}}{\kappa_f+\alpha_f}\right] < 0 \\
\Rightarrow \lim_{\mu_f\to 1}\phi(-z) &= -\frac{z\lim_{\mu_f\to 1}\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(\sigma-1)\left(\kappa_f+\alpha_f\right)} = -\frac{2z\widehat{\eta}^{-z}}{(\sigma-1)\left(\kappa_f+\alpha_f\right)} < 0
\end{aligned}$$

The total effect of foreign start-up costs on $\overline{\varphi}_f$ is:

$$\frac{d\widetilde{\varphi}_f}{d\kappa_f} = \frac{\partial\widetilde{\varphi}_f}{\partial\kappa_f} + \left(\frac{\partial\widetilde{\varphi}_f}{\partial\tau^*}\right)\left(\frac{\partial\tau^*}{\partial\kappa_f}\right)$$

Note that:

$$\begin{aligned}
\frac{\partial x_d^*}{\partial\kappa_f} &= \frac{\psi}{a}\left(\Phi^*\right)^{\frac{1}{a}-1}\left(\frac{\partial\Phi^*}{\partial\kappa_f}\right) = \frac{x_d^*\phi(-z)}{a\left(1+\tau^*\right)^a\Phi^*} \\
\frac{\partial x_d^*}{\partial\tau^*} &= \frac{\psi}{a}\left(\Phi^*\right)^{\frac{1}{a}-1}\left(\frac{\partial\Phi^*}{\partial\tau^*}\right) = \frac{-x_d^*\left(\eta_2^{-z}+\eta_3^{-z}\right)}{\left(1+\tau^*\right)^{a+1}\Phi^*}
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{\partial\widetilde{\varphi}_f}{\partial\kappa_f} &= \nu\left(1+\tau^*\right)\left[x_d^*\frac{\partial H}{\partial\kappa_f} + \frac{\partial x_d^*}{\partial\kappa_f}H\right] = \nu\left(1+\tau^*\right)x_d^*\left[\frac{\partial H}{\partial\kappa_f} + \frac{\phi(-z)H}{a\left(1+\tau^*\right)^a\Phi^*}\right] \\
&= \frac{\nu x_d^*}{a\left(1+\tau^*\right)^{a-1}\Phi^*}\left[a\left(1+\tau^*\right)^a\Phi^*\frac{\partial H}{\partial\kappa_f} + \phi(-z)H\right]
\end{aligned}$$

where:

$$\begin{aligned}
\frac{\partial H}{\partial \kappa_f} &= \frac{1}{\sigma-1}\left(\frac{\eta_2^{-z}+\eta_3^{-z}}{\eta_2^{-a}+\eta_3^{-a}}\right)^{\frac{1}{\sigma-1}-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi(-z)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi(-a)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)^2}\right] \\
&= \frac{H}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi(-z)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi(-a)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right]
\end{aligned}$$

And:

$$\begin{aligned}
\frac{\partial \widetilde{\varphi}_f}{\partial \tau^*} &= \nu H\left[(1+\tau^*)\frac{\partial x_d^*}{\partial \tau^*}+x_d^*\right] = \nu H\left[x_d^* - \frac{x_d^*\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau^*)^a\Phi^*}\right] \\
&= \frac{\nu H x_d^*}{(1+\tau^*)^a\Phi^*}\left[(1+\tau^*)^a\Phi^*-\left(\eta_2^{-z}+\eta_3^{-z}\right)\right] = \frac{\nu H\left(1+\eta_1^{-z}\right)x_d^*}{\Phi^*} > 0
\end{aligned}$$

And:

$$\frac{\partial \tau^*}{\partial \kappa_f} = \frac{-\phi(-z)}{(1+\tau^*)^a\Gamma_{\tau^*}} < 0$$

Combining this information to calculate the total effect of foreign startup costs yields:

$$\begin{aligned}
\frac{d\widetilde{\varphi}_f}{d\kappa_f} &= \frac{\nu x_d^*}{a(1+\tau^*)^{a-1}\Phi^*}\left[a(1+\tau^*)^a\Phi^*\frac{\partial H}{\partial \kappa_f}+\phi(-z)H\right] - \frac{\nu H\left(1+\eta_1^{-z}\right)\phi(-z)x_d^*}{(1+\tau^*)^a\Gamma_{\tau^*}\Phi^*} \\
&= \frac{\nu x_d^*}{a(1+\tau^*)^{a-1}\Phi^*}\left[a(1+\tau^*)^a\Phi^*\frac{\partial H}{\partial \kappa_f}+\phi(-z)H - \frac{aH\left(1+\eta_1^{-z}\right)\phi(-z)}{(1+\tau^*)\Gamma_{\tau^*}}\right] \\
&= \frac{\nu x_d^* H}{a(1+\tau^*)^{a-1}\Phi^*}\left\{\frac{a(1+\tau^*)^a\Phi^*}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi(-z)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi(-a)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right]\right. \\
&\qquad \left. +\phi(-z) - \frac{a\left(1+\eta_1^{-z}\right)\phi(-z)}{(1+\tau^*)\Gamma_{\tau^*}}\right\}
\end{aligned}$$

So $0 < \frac{d\widetilde{\varphi}_f}{d\kappa_f}$ iff:

$$0 < \frac{a(1+\tau^*)^a\Phi^*}{\sigma-1}\left[\frac{\left(\eta_2^{-a}+\eta_3^{-a}\right)\phi(-z)-\left(\eta_2^{-z}+\eta_3^{-z}\right)\phi(-a)}{\left(\eta_2^{-a}+\eta_3^{-a}\right)\left(\eta_2^{-z}+\eta_3^{-z}\right)}\right] + \phi(-z)\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right] \equiv M$$

Recall that $\widehat{\eta} \equiv \eta_3 = \lim_{\mu_f \to 1}\eta_2$.
Then:

$$\begin{aligned}
\lim_{\mu_f\to 1} M &= \frac{a(1+\tau^*)^a\Phi^*}{\sigma-1}\left[\frac{2\widehat{\eta}^{-z}\left[\frac{2a\widehat{\eta}^{-a}}{(\sigma-1)(\kappa_f+\alpha_f)}\right]-2\widehat{\eta}^{-a}\left[\frac{2z\widehat{\eta}^{-z}}{(\sigma-1)(\kappa_f+\alpha_f)}\right]}{4\widehat{\eta}^{-a-z}}\right] \\
&\qquad -\frac{2z\widehat{\eta}^{-z}}{(\sigma-1)(\kappa_f+\alpha_f)}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right] \\
&= \frac{1}{(\sigma-1)(\kappa_f+\alpha_f)}\left\{a(1+\tau^*)^a\Phi^* - 2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\right\} \\
&= \frac{1}{(\sigma-1)(\kappa_f+\alpha_f)}\left\{a(1+\tau^*)^a\left(1+\eta_1^{-z}\right)+2a\widehat{\eta}^{-z}-2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\right\}
\end{aligned}$$

By the Proof of Proposition 3, in equilibrium:

$$\Gamma(\tau^*) = 0 \quad \Leftrightarrow \quad 1+\eta_1^{-z} = \frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau^*)^a(\tau^* a-\tau^*-1)} \quad \Leftrightarrow \quad (1+\tau^*)^a\left(1+\eta_1^{-z}\right) = \frac{\eta_2^{-z}+\eta_3^{-z}}{\tau^* a-\tau^*-1} \qquad (4)$$

Using this substitution means that in equilibrium:

$$\begin{aligned}\lim_{\mu_f\to 1} M &= \frac{1}{(\sigma-1)(\kappa_f+\alpha_f)}\left\{a\left[\lim_{\mu_f\to 1}\frac{\eta_2^{-z}+\eta_3^{-z}}{\tau^* a-\tau^*-1}\right]+2a\widehat{\eta}^{-z}-2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\right\}\\ &= \frac{1}{(\sigma-1)(\kappa_f+\alpha_f)}\left\{\frac{2a\widehat{\eta}^{-z}}{\tau^* a-\tau^*-1}+2a\widehat{\eta}^{-z}-2z\widehat{\eta}^{-z}\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\right\}\\ &= \frac{2\widehat{\eta}^{-z}}{(\sigma-1)(\kappa_f+\alpha_f)}\left\{\frac{a(a-1)\tau^*}{\tau^* a-\tau^*-1}-z\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]\right\}\end{aligned}$$

If we use the derivation of Γ_{τ^*} from the Proof of Proposition 4 and the information from equation (4), we can see that in equilibrium:

$$\begin{aligned}\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}} &= \frac{-a\left(1+\eta_1^{-z}\right)}{(a-1)(1+\tau^*)\left(1+\eta_1^{-z}\right)+\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau^*)^a}}\\ &= \frac{-a\left[\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau^*)^a(\tau^* a-\tau^*-1)}\right]}{(a-1)(1+\tau^*)\left[\frac{\eta_2^{-z}+\eta_3^{-z}}{(1+\tau^*)^a(\tau^* a-\tau^*-1)}\right]+\frac{a\left(\eta_2^{-z}+\eta_3^{-z}\right)}{(1+\tau^*)^a}}\\ &= \frac{-a}{(a-1)(1+\tau^*)+a(\tau^* a-\tau^*-1)}\\ &= \frac{-a}{(a+1)(a-1)\tau^*-1}\end{aligned}$$

So:

$$0<\lim_{\mu_f\to 1} M \quad\Leftrightarrow\quad z\left[1-\frac{a\left(1+\eta_1^{-z}\right)}{(1+\tau^*)\Gamma_{\tau^*}}\right]=z(a-1)\left[\frac{(a+1)\tau^*+1}{(a+1)(a-1)\tau^*-1}\right]<\frac{a(a-1)\tau^*}{\tau^* a-\tau^*-1}$$

Note that by equation (4), in equilibrium:

$$0<\tau^* a-\tau^*-1=(a-1)\tau^*-1<(a+1)(a-1)\tau^*-1=\left(a^2-1\right)\tau^*-1$$

So:

$$\begin{aligned}0<\lim_{\mu_f\to 1} M \quad&\Leftrightarrow\quad z\left[(a+1)\tau^*+1\right](\tau^* a-\tau^*-1)<a\tau^*\left[\left(a^2-1\right)\tau^*-1\right]\\ &\Leftrightarrow\quad z\left\{\left[(a+1)\tau^*+1\right](\tau^* a-\tau^*-1)-\tau^*\left[\left(a^2-1\right)\tau^*-1\right]\right\}<(a-z)\tau^*\left[\left(a^2-1\right)\tau^*-1\right]\\ &\Leftrightarrow\quad -z(\tau^*+1)<(\sigma-1)\tau^*\left[\left(a^2-1\right)\tau^*-1\right]\end{aligned}$$

This holds in equilibrium. QED.

Proof of Proposition 6

As shown in the Proof of Proposition 1, the equilibrium revenue of a foreign firm in industry j is:

$$r_{fj}^*(\varphi)=\left[\frac{\varphi}{\rho(1+\tau_j)}\right]^{\sigma-1}Q_j P_j^{\sigma}=\left[\frac{\varphi}{\rho(1+\tau_j)}\right]^{\sigma-1}Lw_j P_j\left(x_{dj}^*\right)^{\sigma-1}=\frac{zLw_j\left(x_{dj}^*\right)^z\varphi^{\sigma-1}}{ab^a(1+\tau_j)^{\sigma-1}\Phi_j^*}$$

The total effect of foreign start-up costs on foreign revenue will be:

$$\frac{dr_{fj}^*(\varphi)}{d\kappa_{fj}}=\frac{\partial r_{fj}^*(\varphi)}{\partial\kappa_{fj}}+\frac{\partial r_{fj}^*(\varphi)}{\partial\tau_j^*}\left(\frac{\partial\tau_j^*}{\partial\kappa_{fj}}\right)$$

Note that:

$$\begin{aligned}
\frac{\partial r^*_{fj}(\varphi)}{\partial \kappa_{fj}} &= \frac{zLw_j\varphi^{\sigma-1}}{ab^a(1+\tau_j)^{\sigma-1}}\left[\frac{\Phi^*_j z\left(x^*_{dj}\right)^{z-1}\left(\frac{\partial x^*_{dj}}{\partial \kappa_{fj}}\right)-\left(x^*_{dj}\right)^z\left(\frac{\partial \Phi^*_j}{\partial \kappa_{fj}}\right)}{\left(\Phi^*_j\right)^2}\right] \\
&= \frac{zLw_j\varphi^{\sigma-1}}{ab^a(1+\tau_j)^{\sigma-1}\left(\Phi^*_j\right)^2}\left[\Phi^*_j z\left(x^*_{dj}\right)^{z-1}\left(\frac{x^*_{dj}}{a\Phi^*_j}\right)\left(\frac{\partial \Phi^*_j}{\partial \kappa_{fj}}\right)-\left(x^*_{dj}\right)^z\left(\frac{\partial \Phi^*_j}{\partial \kappa_{fj}}\right)\right] \\
&= \frac{zLw_j\varphi^{\sigma-1}\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma-1}\left(\Phi^*_j\right)^2}\left(\frac{z-a}{a}\right)\left(\frac{\partial \Phi^*_j}{\partial \kappa_{fj}}\right) = -\frac{zLw_j\varphi^{\sigma-1}\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma-1}\left(\Phi^*_j\right)^2}\left(\frac{\sigma-1}{a}\right)\left(\frac{\partial \Phi^*_j}{\partial \kappa_{fj}}\right)
\end{aligned}$$

where:

$$\frac{\partial \Phi^*_j}{\partial \kappa_{fj}} = -z(1+\tau_j)^{-a}\left[\eta_{2j}^{z-1}\left(\frac{\partial \eta_{2j}}{\partial \kappa_{fj}}\right)+\eta_{3j}^{z-1}\left(\frac{\partial \eta_{3j}}{\partial \kappa_{fj}}\right)\right]<0 \quad \Rightarrow \quad \frac{\partial r^*_{fj}(\varphi)}{\partial \kappa_{fj}}>0$$

Also:

$$\begin{aligned}
\frac{\partial r^*_{fj}(\varphi)}{\partial \tau_j} &= \frac{zLw_j\varphi^{\sigma-1}}{ab^a}\left\{\frac{(1+\tau_j)^{\sigma-1}\Phi^*_j z\left(x^*_{dj}\right)^{z-1}\left(\frac{\partial x^*_{dj}}{\partial \tau_j}\right)-\left(x^*_{dj}\right)^z\left[(1+\tau_j)^{\sigma-1}\left(\frac{\partial \Phi^*_j}{\partial \tau_j}\right)+(\sigma-1)(1+\tau_j)^{\sigma-2}\Phi^*_j\right]}{(1+\tau_j)^{2(\sigma-1)}\left(\Phi^*_j\right)^2}\right\} \\
&= \frac{zLw_j\varphi^{\sigma-1}}{ab^a(1+\tau_j)^{\sigma}\left(\Phi^*_j\right)^2}\left\{(1+\tau_j)\Phi^*_j z\left(x^*_{dj}\right)^{z-1}\left(\frac{x^*_{dj}}{a\Phi^*_j}\right)\left(\frac{\partial \Phi^*_j}{\partial \tau_j}\right)-\left(x^*_{dj}\right)^z\left[(1+\tau_j)\left(\frac{\partial \Phi^*_j}{\partial \tau_j}\right)+(\sigma-1)\Phi^*_j\right]\right\} \\
&= \frac{zLw_j\varphi^{\sigma-1}\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma}\left(\Phi^*_j\right)^2}\left[(1+\tau_j)\left(\frac{z-a}{a}\right)\left(\frac{\partial \Phi^*_j}{\partial \tau_j}\right)-(\sigma-1)\Phi^*_j\right] \\
&= -\frac{z(\sigma-1)Lw_j\varphi^{\sigma-1}\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma}\left(\Phi^*_j\right)^2}\left[\left(\frac{1+\tau_j}{a}\right)\left(\frac{\partial \Phi^*_j}{\partial \tau_j}\right)+\Phi^*_j\right] \\
&= -\frac{z(\sigma-1)Lw_j\varphi^{\sigma-1}\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma}\left(\Phi^*_j\right)^2}\left[\Phi^*_j-(1+\tau_j)^{-a}\left(\eta_{2j}^{-z}+\eta_{3j}^{-z}\right)\right] \\
&= -\frac{z(\sigma-1)Lw_j\varphi^{\sigma-1}\left(1+\eta_{1j}^{-z}\right)\left(x^*_{dj}\right)^z}{ab^a(1+\tau_j)^{\sigma}\left(\Phi^*_j\right)^2}<0
\end{aligned}$$

Finally, by the Proof of Proposition 4: $\frac{\partial \tau^*_j}{\partial \kappa_{fj}}<0$. So $\frac{dr^*_{fj}(\varphi)}{d\kappa_{fj}}>0$. QED